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ABSTRACT

Many ancient paintings, in particular frescoes, have some parts ruined by time and events. Sometimes one or more non-negligible regions are completely lost, leaving a blank that is called by restaurateurs a 'lacuna'. The general restoration philosophy adopted in these cases is to paint the interior part of the lacuna with an achromatic or non-neutral uniform color carefully selected in order to minimize its overall perception. In this paper, we present a computational model, based on a well-established variational theory of color perception, that may facilitate the job of a restaurateur by providing both achromatic and non-neutral colors which minimize the local contrast with the surrounding parts of the fresco.

Keywords: Perceived contrast, frescoes, lacunae

1. INTRODUCTION

In the paper,[1] the authors have introduced a variational model for color correction of digital images based on some important phenomenological features of the Human Visual System (HVS from now on). These properties are: *color constancy*, the HVS robustness about color perception with respect to changes of illumination; *local contrast enhancement*, put in evidence e.g. by the famous Mach bands effect;[2] *adaptation to the average radiance* level of a visual scene, and *Weber-Fechner's law* about contrast intensity changes.

The authors shown that there is only one class of energy functionals which can fulfill all these properties at once: they are characterized by the simultaneous presence of two opposing terms. The first term can be interpreted as the (inverse) perceived local contrast and the second as an adjustment to the average and the original intensity levels. The argmin of such a functional is an image which has the property to optimally conjugate local contrast enhancement and adherence to the average intensity level, without departing too much from the original image.

In this paper, we take inspiration from[1] to build a novel computational method which we will apply on problem of tone selection to fill lacunae in fresco restoration. The link between the above quoted paper and this problem relies in the fact that they are both related to the concept of perceived contrast, as we are going to explain.

Many ancient paintings, in particular frescoes, may lack of entire parts due to natural time degradation of their pigments or other, possibly man-caused, events. These blanks are called lacunae and their area may be so extended that they can be perceptively non-negligible. One of the roles of a restaurateur is then to fill lacunae with the aim to make it as least perceptible as possible **INTRODURRE CITAZIONE**. In general, a uniform achromatic color is preferred to a structured patter and to a non-neutral color, but some exceptions are possible **INTRODURRE CITAZIONE**.

In the following section we will briefly recall the variational model,[1] then, in Section 6 we will describe our proposal to help the task of lacunae filling and the corresponding tests in Section 7. Finally, we will conclude the paper in Section 8 with some further considerations about our method.

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2. PERCEPTUALLY INSPIRED ENERGY FUNCTIONAL

The purpose of the article[1] is to find a suitable energy functional whose argmin is a *perceptually inspired* image, i.e. an image closer to human perception of the scene represented in a digital image. This energy will result from a balance between the contrast enhancement and the adjustment to the average intensity level and the original image values.

From now on, a discrete RGB image will be considered as a function $\vec{I} : \Omega \to [0,1]^3, \vec{I}(x) = (I_R(x), I_G(x), I_B(x))$ where $\Omega = \{1,\ldots,N\} \times \{1,\ldots,M\} \subset \mathbb{Z}^2$ and $I_k(x)$ is the value of the pixel $x \in \Omega$ in the chromatic channel $k \in \{R,G,B\}$.

The model proposed attempts to be compatible with the following phenomenological characteristics of the HVS observed experimentally: local contrast enhancement, color constancy, Weber's law and visual adaptation to the average luminance level.

The local contrast enhancement has the aim of improving edge perception because the biggest part of the information encoded inside an image is brought by the edges. This operation is local and influenced by the distribution of intensity around a point.

To include these characteristics in the energy functional, we consider a contrast function $\overline{c} : [0,1]^2 \to \mathbb{R}$ continuous and symmetric in $(a,b) \in [0,1]^2$. The symmetry implies that $\overline{c}$ is a function of $\min(a,b)$ and $\max(a,b)$, moreover $\overline{c}$ has to be decreasing with respect to $\min(a,b)$ and increasing with respect to $\max(a,b)$.

The maximization of a contrast function performs the desired enhancement, so in order to get the same result by the minimization of a functional we have to use an inverse contrast function, that is continuous and symmetric but decreasing with respect to $\min(a,b)$ and increasing with respect to $\max(a,b)$. The locality of the contrast term is accomplished by a symmetric weighting function $w : \Omega \times \Omega \to \mathbb{R}^+$. For simplicity, we assume that the kernel is normalized, i.e.

$$\sum_{y\in\Omega} w(x,y) = 1 \qquad \forall x \in \Omega. \tag{1}$$

By the preceding analysis, the contrast term has the form

$$C_w(I) = \sum_{x\in\Omega}\sum_{y\in\Omega} w(x,y) c(I(x), I(y)). \tag{2}$$

Another HVS feature taken into account is color constancy, which is the ability to recognize colors of a scene under different illuminants. Some pictures show a tint of a particular color that is usually unwanted and affects the whole image evenly: this phenomenon is called color cast and its effect is partially removed by the HVS.

We can notice that the reduction of the color cast in an image is linked to contrast enhancement. As a matter of fact all images that present color cast are characterized by a channel which has the standard deviation unusually higher than the others, and contrast enhancement spreads the intensity values in each channel so it uniforms the values of the standard deviations.

This ability of the HVS suggests that visual perception is not affected by an overall variation in the intensity. To translate this principle in mathematical terms, we impose that the contrast function, and consequently the inverse contrast function, is homogeneous (of degree 0). A function c is said to be homogeneous of degree $n \in \mathbb{Z}$ if $\forall\lambda > 0$ one has:

$$c(\lambda a, \lambda b) = \lambda^n c(a,b).$$

If c is homogeneous, then:

$$c(a,b) = b^0 c\left(\frac{a}{b}, 1\right) = c\left(\frac{a}{b}, 1\right),$$

so it is function of the ratio $\frac{\min(a,b)}{\max(a,b)}$. An inverse contrast function that satisfies this additional assumption can be written as a monotone non-decreasing function of $\frac{\min(a,b)}{\max(a,b)}$. In conclusion, the contrast term has the form

$$C_w^{\varphi}(I) = \sum_{x\in\Omega}\sum_{y\in\Omega} w(x,y)\varphi\left(\frac{\min(I(x),I(y))}{\max(I(x),I(y))}\right), \tag{3}$$

where φ is a non-decreasing differentiable function.

This type of function is also in agreement with Weber's law, which can be written as follows:

$$K = \frac{\Delta I}{I_0} = \frac{I_1 - I_0}{I_0} = \frac{I_1}{I_0} - 1 \tag{4}$$

where K is a constant, I_1 is a just noticeable impulse over a background of intensity I_0.

This law states that the minimum perceived contrast is a function of the ratio between the two intensities and this is compatible with what we have assumed for the inverse contrast function c.

Among all possible choices of φ, the authors of[1] chose the following expressions: $\frac{1}{4}\mathrm{id}$, $\frac{1}{4}\log$, $-\frac{1}{4}\mathcal{M}$ where $\mathcal{M}(x) = \frac{1-x}{1+x}$. The function $\mathcal{M}$ is of particular interest, in fact it coincides with the so-called, Michelson's definition of contrast:

$$\mathcal{M}\left(\frac{\min(I(x), I(y))}{\max(I(x), I(y))}\right) = \frac{1 - \frac{\min(I(x),I(y))}{\max(I(x),I(y))}}{1 + \frac{\min(I(x),I(y))}{\max(I(x),I(y))}} = \frac{|I(x) - I(y)|}{I(x) + I(y)}.$$

The last feature that should be simulated is visual adaptation: to prevent saturation or the impossibility to perceive the details, the HVS shifts the range of perceivable light intensities around the average luminance of the scene. It is easy to incorporate this knowledge in the dispersion term of the energy functional that measures the distance of an image from the middle gray and from the original image. If we call $d : [0,1]\times]0,1] \to \mathbb{R}^+$ a continuous function such that $d(a,b) = 0$ if and only if $a = b$, we can express the dispersion term in the form

$$D^d_{\alpha,\beta}(I) = \alpha \sum_{x\in\Omega} d\left(I(x), \frac{1}{2}\right) + \beta \sum_{x\in\Omega} d\left(I(x), I_0(x)\right) \tag{5}$$

where $\frac{1}{2}$ is the value of the middle gray in the normalized range $[0,1]$ and $\alpha, \beta > 0$ are parameters that balance the importance of the two components of the term.

We could choose an arbitrary distance, like the Euclidean one, as the d function, but there is an additional constraint on this function given by the dimensional coherence with the contrast term of degree 0. We will deepen this point in the next section, where we will show that the contrast term and the function $\mathcal{E}(a,b) = a\log\left(\frac{a}{b}\right) - (a-b)$ have dimensionally coherent gradients. By now it is sufficient to notice that the function $\mathcal{E}(a,b)$ satisfies the hypothesis. In fact $\forall a \in [0,1]$ $f_a(s) = a\log\left(\frac{a}{s}\right) - (a-s)$ has a unique minimum in $s = a$ and $f_a(a) = 0$. The form of $\mathcal{E}$ agrees with the entropic dispersion, according to this interpretation $\mathcal{E}$ measures the disorder around $\frac{1}{2}$ and around the original image.

By putting together the equations (3) and (5), we get the complete general form of a perceptually inspired energy functional

$$E(I) = C^\varphi_w(I) + D^d_{\alpha,\beta}(I) = \sum_{x\in\Omega}\sum_{y\in\Omega} w(x,y)\varphi\left(\frac{\min(a,b)}{\max(a,b)}\right) + \alpha\sum_{x\in\Omega} d\left(I(x), \frac{1}{2}\right) + \beta\sum_{x\in\Omega} d\left(I(x), I_0(x)\right). \tag{6}$$

The interpretation of this functional is straightforward: the contrast enhancement leads towards the minimization of the term C^φ_w, but at the same time it brings the image far from the original one increasing the value of the term $D^d_{\alpha,\beta}$. The final image is obtained by finding the equilibrium between these two opposite forces.

3. GRADIENT OF THE ENERGY FUNCTIONAL

Before starting the research of a minimum, it is wise to find the conditions under which a minimum exists. Because of the singularity of the logarithm (present in the entropic dispersion term) in the point 0, we have to exclude all images with null value pixel. From the point of view of human perception setting all null intensities to 1 in an image is a negligible and indiscernible modification. In mathematical terms $\forall\rho > 0$ we define the subset of the space of images

$$\mathcal{F}_\rho = \{I : \Omega \to [0,1], I(x) \geq \rho\ \forall x \in \Omega\} \tag{7}$$

and we show that a minimum of E exists on the set $\mathcal{F}_\rho$.

PROPOSITION 1. *Let* $c\ :]0,1]\times]0,1]\to\mathbb{R}$ *be a continuous function,* $\forall\rho>0$ *there is a minimum of the energy functional* $E(I)=C_w(I)+D^{\mathcal{E}}_{\alpha,\beta}$ *on* $\mathcal{F}_\rho$ *where* $C_w(I)$ *has the general form of (2).*

Proof. We can identify an image whose support is $\Omega=\{1,\dots,N\}\times\{1,\dots,M\}$ *with a vector* $u\in\mathbb{R}^{NM}$*. With the same identification* $\mathcal{F}_\rho=\{u\in\mathbb{R}^{NM}|u_i\in[\rho,1]\ \forall i=1,\dots,NM\}$ *is a compact set of* $\mathbb{R}^{NM}$*.* E *is trivially continuous and it is well-known that a continuous function on a compact set always admits a minimum.* □

In order to find the minimum we would like to use a gradient descent algorithm, but this is possible only in the case of a differentiable energy functional, because otherwise it is not possible to define a gradient. It is easy to notice that E cannot be differentiable independently of the choice of the functions w,φ and d because the function $\frac{\min(a,b)}{\max(a,b)}$ is not differentiable. The non-differentiability is caused by the hidden presence of the absolute value in this last function, in fact:

$$\min(a,b)=\frac{1}{2}(a+b-|a-b|)\qquad \max(a,b)=\frac{1}{2}(a+b+|a-b|).$$

To solve the problem we will use a regularized version of the absolute value $A(z)=|z|$ to define a differentiable perceptually inspired energy E.

DEFINITION 2. *Given* $\epsilon>0$*, we call* $A_\epsilon(z)$ *a "nice regularizatio" of* $A(z)$ *if* $A_\epsilon(z)\geq 0$ *is of class* $\mathcal{C}^1$ *(with the derivative denoted by* $s_\epsilon(z)$*), convex, pair and*
1. $\forall z\in\mathbb{R}\ A_\epsilon(z)\leq|z|$ *and* $Q_{1,\epsilon}(z)=A_\epsilon(z)-|z|\xrightarrow[\epsilon\to 0]{}0$ *uniformly in* $[-1,1]$*;*
2. $\forall z\in[-1,1]\ s_\epsilon(z)\leq 1,\forall z\in\mathbb{R}\ s_\epsilon(z)\xrightarrow[\epsilon\to 0]{}\operatorname{sign}(z)$ *and* $Q_{2,\epsilon}(z)=A_\epsilon(z)-zs_\epsilon(z)\xrightarrow[\epsilon\to 0]{}0$ *uniformly in* $[-1,1]$*.*

PROPOSITION 3. *Given* $\epsilon>0$*,* $A_\epsilon(z)=z\frac{\arctan(z/\epsilon)}{\arctan(1/\epsilon)}-\frac{\epsilon}{2\arctan(1/\epsilon)}\log\left(1+\frac{z^2}{\epsilon^2}\right)$ *is a nice regularisation of* $A(z)$*. Its derivative is* $s_\epsilon(z)=\frac{\arctan(z/\epsilon)}{\arctan(1/\epsilon)}$*.* The Proposition is proved in the paper.? Now we can define the regularized versions of $\min$ and $\max$ by using a nice regularisation of $A(z)$, denoted by $A_\epsilon(z)$, as

$$\min\nolimits_\epsilon(a,b)=\tfrac{1}{2}(a+b-A_\epsilon(a-b))\qquad \max\nolimits_\epsilon(a,b)=\tfrac{1}{2}(a+b+A_\epsilon(a-b)).$$

The resulting differentiable contrast functionals that we will analyze are

$$C^{\mathrm{id}}_{w,\epsilon}=\frac{1}{4}\sum_{x\in\Omega}\sum_{y\in\Omega}w(x,y)\frac{\min_\epsilon(I(x),I(y))}{\max_\epsilon(I(x),I(y))},\tag{8}$$

$$C^{\log}_{w,\epsilon}=\frac{1}{4}\sum_{x\in\Omega}\sum_{y\in\Omega}w(x,y)\log\left(\frac{\min_\epsilon(I(x),I(y))}{\max_\epsilon(I(x),I(y))}\right),\tag{9}$$

$$C^{-\mathcal{M}}_{w,\epsilon}=-\frac{1}{4}\sum_{x\in\Omega}\sum_{y\in\Omega}w(x,y)\frac{A_\epsilon(I(x)-I(y))}{I(x)+I(y)}.\tag{10}$$

$$\tag{11}$$

In order to proceed in doing the calculation of the Gteaux gradients of these functionals, we need a preliminary lemma.

LEMMA 4. *Let* $w:\Omega^2\to\mathbb{R}$ *be a symmetric function and* $F\ :]0,+\infty[^2\to\mathbb{R}$ *be a differentiable symmetric function, if we denote* $F_1(a,b)=\frac{\partial F}{\partial a}(a,b)$*, then the Gteaux gradient of*

$$C(I)=\sum_{x\in\Omega}\sum_{y\in\Omega}w(x,y)F(I(x),I(y))\tag{12}$$

can be written as

$$\nabla C(I)(x)=2\sum_{y\in\Omega}w(x,y)F_1(I(x),I(y)).\tag{13}$$

Proof. Firstly we remark that the symmetry of F *implies* $F_2(b,a)=\frac{\partial F}{\partial a}(b,a)=\frac{\partial F}{\partial a}(a,b)=F_1(a,b)$*. The Gteaux differential of* $C(I)$ *is defined as the function* $\delta C:\Omega^2\to\mathbb{R}$ *such that* $\delta C(\delta I)=\psi'_{\delta I}(0)$ *where*

$$\psi_{\delta I}(t)=\sum_{x\in\Omega}\sum_{y\in\Omega}w(x,y)F(I(x)+t\delta I(x),I(y)+t\delta I(y)).$$

From the definition we get

$$\delta C(I).\delta I = \sum_{x\in\Omega}\sum_{y\in\Omega} w(x,y)F_1(I(x),I(y))\delta I(x) + \sum_{x\in\Omega}\sum_{y\in\Omega} w(x,y)F_2(I(x),I(y))\delta I(y)$$

and interchanging x and y in the second sum, remembering that $F_1(a,b) = F_2(b,a)$ we obtain

$$\delta C(I).\delta I = 2\sum_{x\in\Omega}\sum_{y\in\Omega} w(x,y)F_1(I(x),I(y))\delta I(x) = \langle 2\sum_{y\in\Omega} w(x,y)F_1(I(x),I(y)), \delta I(x)\rangle.$$

Since $\nabla C(I)$ is the only function such that $\delta C(I).\delta I = \langle \nabla C(I), \delta I(x)\rangle$, the Lemma is proved. □

PROPOSITION 5. *Given $\epsilon > 0$, let $A_\epsilon(z)$ be a nice regularization of $A(z)$, then*

$$\nabla C^{\text{id}}_{w,\epsilon}(I)(x) = -\frac{1}{2}\sum_{y\in\Omega} w(x,y)\frac{I(y)}{\max_\epsilon(I(x),I(y))^2} s_\epsilon(I(x)-I(y)) + S_\epsilon \tag{14}$$

$$\nabla C^{\log}_{w,\epsilon}(I)(x) = -\frac{1}{2}\sum_{y\in\Omega} w(x,y)\frac{1}{I(x)} s_\epsilon(I(x)-I(y)) + S_\epsilon \tag{15}$$

$$\nabla C^{-\mathcal{M}}_{w,\epsilon}(I)(x) = -\sum_{y\in\Omega} w(x,y)\frac{I(y)}{(I(x)+I(y))^2} s_\epsilon(I(x)-I(y)) + S_\epsilon \tag{16}$$

where $S_\epsilon = \mathcal{O}(Q_{1,\epsilon}(I(x)-I(y)) + Q_{2,\epsilon}(I(x)-I(y)))$.

Proof. In all the cases we can utilize Lemma 4 with the following F functions:

$$\begin{aligned}
F^{id}(a,b) = \frac{1}{4}\frac{\min_\epsilon(a,b)}{\max_\epsilon(a,b)} &\Rightarrow F_1^{id}(a,b) = \frac{1}{8}\frac{A_\epsilon(a-b)-(a+b)s_\epsilon(a-b)}{\max_\epsilon(a,b)^2},\\
F^{\log}(a,b) = \frac{1}{4}\log\left(\frac{\min_\epsilon(a,b)}{\max_\epsilon(a,b)}\right) &\Rightarrow F_1^{\log}(a,b) = \frac{1}{8}\frac{A_\epsilon(a-b)-(a+b)s_\epsilon(a-b)}{\min_\epsilon(a,b)\max_\epsilon(a,b)},\\
F^{-\mathcal{M}}(a,b) = -\frac{1}{4}\frac{A_\epsilon(a-b)}{a+b} &\Rightarrow F_1^{-\mathcal{M}}(a,b) = \frac{1}{4}\frac{A_\epsilon(a-b)-(a+b)s_\epsilon(a-b)}{(a+b)^2}.
\end{aligned}$$

We show the details of the proof just for $C^{id}_{w,\epsilon}$ because the other cases follow the same reasoning. We can rewrite F_1^{id} as

$$F_1^{id}(a,b) = -\frac{1}{4}\frac{b}{\max_\epsilon(a,b)^2}s_\epsilon(a-b) + \frac{1}{8}\frac{Q_{2,\epsilon}(a-b)}{\max_\epsilon(a,b)^2}$$

so by Lemma 4 we have the desired result

$$\begin{aligned}
\nabla C^{id}_{w,\epsilon}(I)(x) &= -\frac{1}{2}\sum_{y\in\Omega} w(x,y)\left(\frac{I(y)}{\max_\epsilon(I(x),I(y))^2}s_\epsilon(I(x)-I(y)) - \frac{1}{2}\frac{Q_{2,\epsilon}(I(x)-I(y))}{\max_\epsilon(I(x),I(y))^2}\right)\\
&= -\frac{1}{2}\sum_{y\in\Omega} w(x,y)\frac{I(y)}{\max_\epsilon(I(x),I(y))^2}s_\epsilon(I(x)-I(y)) + S'_\epsilon
\end{aligned}$$

where $S_\epsilon = \mathcal{O}(Q_{2,\epsilon}(I(x)-I(y)))$. □

We have just found the gradients associated with the studied contrast terms, we now go through the calculation of the gradient of the entropic dispersion term.

PROPOSITION 6. *Given $\alpha, \beta > 0$, let*

$$D^{\mathcal{E}}_{\alpha,\beta}(I) = \alpha\sum_{x\in\Omega}\left(\frac{1}{2}\log\frac{1}{2I(x)} - \left(\frac{1}{2} - I(x)\right)\right) + \beta\sum_{x\in\Omega}\left(I_0(x)\log\frac{I_0(x)}{I(x)} - (I_0(x)-I(x))\right) \tag{17}$$

be the entropic dispersion term, then

$$\nabla D^{\mathcal{E}}_{\alpha,\beta}(I)(x) = \alpha\left(1 - \frac{1}{2I(x)}\right) + \beta\left(1 - \frac{I_0(x)}{I(x)}\right). \tag{18}$$

Proof. The result is obtained simply by applying the definition of the Gteaux gradient. □

Now it is possible to explain the choice of the function $\mathcal{E}$ in the dispersion term. As a matter of fact the gradient of every analyzed contrast term has a degree of homogeneity -1 with respect to $I(x)$, and to keep a dimensional coherence it is necessary to have the same degree also in the gradient of the dispersion term. The entropic dispersion term fulfills this requirement, while it is not the case for the Euclidean distance, for example.

We can conclude the section by remarking that by linearity we have

$$\nabla E(I) = \nabla C^{\varphi}_{w,\epsilon}(I) + \nabla D^{\mathcal{E}}_{\alpha,\beta}.$$

With the complete expression of the Gteaux gradient of the perceptually inspired energy, we can proceed to the application of a gradient descent algorithm to find the minimum.

4. GRADIENT DESCENT ALGORITHM

Like in many complex problems, also in this case the equation $\nabla E(I) = 0$ cannot be solved analytically, but the research of a minimum can only be accomplished by numerical means. The algorithm proposed in the paper follows a semi-implicit discrete gradient descent strategy with respect to $\log I$. It has been proven[?] that this scheme converges to the same minimum of a regular gradient descent algorithm, the only difference is the reduced speed of convergence. The motivation for this choice comes from dimensional considerations, like the ones that have led to the selection of the entropic dispersion term. In mathematical terms we can write the scheme in the form

$$\partial_t \log I = -\nabla E(I) \Rightarrow \partial_t I = -I\nabla E(I). \tag{19}$$

It is possible to see that the second form is dimensionally consistent because both the sides of the equation have degree 0 in terms of I. Indeed we have shown in the previous section $\nabla E(I)$ always has dimension -1 with respect to I. We are going to use this formulation to discretize the scheme.

We set a time step $\Delta t > 0$ and we denote by $I^k = I_{k\Delta T}$ $\forall k \in \mathbb{N}$ the image computed at the k^{th} iteration, then the iterative formula obtained by the semi-implicit discretization of (19) is

$$\frac{I^{k+1}(x) - I^k(x)}{\Delta t} = \alpha\left(\frac{1}{2} - I^{k+1}(x)\right) + \beta\left(I_0(x) - I^{k+1}(x)\right) - I^k(x)\nabla C^{\varphi}_{w,\epsilon}(I^k)(x).$$

from which we get the final equation

$$I^{k+1}(x) = \frac{I^k(x) + \Delta t\left(\frac{\alpha}{2} + \beta I_0(x) + \frac{1}{2}R^{\varphi}_{\epsilon,I^k}(x)\right)}{1 + \Delta t(\alpha+\beta)} \tag{20}$$

where $R^{\varphi}_{\epsilon,I^k}(x) = -2I^k\nabla C^{\varphi}_{w,\epsilon}(I^k)(x)$. In doing the effective computation of $R^{\varphi}_{\epsilon,I^k}(x)$ we can ignore the negligible term S_ϵ so the three explicit forms for the different choices of φ are

$$R^{\text{id}}_{\epsilon,I^k}(x) = \sum_{y\in\Omega} w(x,y)\frac{I^k(x)I^k(y)}{\max_\epsilon(I^k(x),I^k(y))^2}s_\epsilon(I^k(x) - I^k(y)) \tag{21}$$

$$R^{\log}_{\epsilon,I^k}(x) = \sum_{y\in\Omega} w(x,y)s_\epsilon(I^k(x) - I^k(y)) \tag{22}$$

$$R^{-\mathcal{M}}_{\epsilon,I^k}(x) = \sum_{y\in\Omega} w(x,y)\frac{2I^k(x)I^k(y)}{(I^k(x)+I^k(y))^2}s_\epsilon(I^k(x) - I^k(y)). \tag{23}$$

In Proposition 1 we have shown that a minimum exists on the compact $\mathcal{F}_\rho$, so in order to use this result we need to ensure that at each iteration the resulting image I_k takes values in the interval $[\rho, 1]$.

PROPOSITION 7. *Let $\rho > 0$, if $I_0 \in \mathcal{F}_\rho$ and $\alpha \geq \frac{1}{1-2\rho} > 0$ then $\forall\, k \in \mathbb{N},\ I^k \in \mathcal{F}_\rho$, i.e. $\rho \leq I^k(x) \leq 1\ \forall\, x \in \Omega$.*

Proof. We can observe that, in all the cases, the function $R^{\varphi}_{\epsilon,I}$ takes the form

$$R^{\varphi}_{\epsilon,I}(x) = \sum_{y\in\Omega} w(x,y) r(I(x), I(y)) s_\epsilon(I(x) - I(y)) \quad \textit{where} \quad 0 \leq r(I(x), I(y)) \leq 1\, \forall\, x, y \in \Omega$$

so

$$|R^{\varphi}_{\epsilon,I}(x)| \leq \sum_{y\in I} w(x,y) r(I(x), I(y)) |s_\epsilon(I(x) - I(y))| \leq \sum_{y\in\Omega} w(x,y) = 1 \quad \forall\, x \in \Omega.$$

Now, by using this fact, we can prove by induction on k that $I^{k+1}(x) \in [\rho, 1]\ \forall x \in \Omega$. We assume $I^k(x) \in [\rho, 1]\ \forall x \in \Omega$ then from the iteration formula (20) we have

$$I^{k+1}(x) \leq \frac{1 + \Delta t(\frac{\alpha}{2} + \beta + \frac{1}{2})}{1 + \Delta t(\alpha + \beta)} \leq \frac{1 + \Delta t(\alpha + \beta)}{1 + \Delta t(\alpha + \beta)} = 1 \quad \textit{because}\ \frac{\alpha}{2} \geq \frac{1}{2}\frac{1}{1-2\rho} > \frac{1}{2},$$

$$I^{k+1}(x) \geq \frac{\rho + \Delta t(\frac{\alpha}{2} + \beta\rho - \frac{1}{2})}{1 + \Delta t(\frac{\alpha-1}{2} + \beta)} \geq \rho\frac{1 + \Delta t(\alpha + \beta)}{1 + \Delta t(\alpha + \beta)} = \rho \quad \textit{because}\ \alpha \geq \frac{1}{1-2\rho} \Rightarrow \frac{\alpha}{2} - \frac{1}{2} \geq \alpha\rho.$$

□

In the real implementation ρ is set to the lowest non-zero intensity value for a pixel, $\rho = \frac{1}{255}$, so we have to pone $\alpha \geq \frac{255}{253}$.

5. COMPUTATIONAL COMPLEXITY REDUCTION OF THE ALGORITHM

Each iteration of the algorithm proposed has a computational complexity of $\mathcal{O}(N^2)$ where N is the number of pixels in the image. Indeed for each pixel $x \in \Omega$ it is necessary to compute the term $R^{\varphi}_{\epsilon,I}(x)$ which requires to sum over all the N pixel in the image. With this high complexity the algorithm can take hours to terminate. The authors propose a way to reduce the complexity to $\mathcal{O}(N \log N)$ by reducing the problem to the application of the FFT (Fast Fourier Transform) algorithm. In order to perform this approximation of $R^{\varphi}_{\epsilon,I}(x)$, we have to assume that the kernel w is a radial function, i.e. it can be expressed as a function of $||x-y||$. If we could write $R^{\varphi}_{\epsilon,I}$ in the following way

$$R^{\varphi}_{\epsilon,I}(x) = \sum_{y\in\Omega} w(||x-y||) f(I(x)) g(I(y)) = f(I(x)) \sum_{y\in\Omega} w(||x-y||) g(I(y)) \tag{24}$$

by separating the dependence between $I(x)$ and $I(y)$, we would have to compute just some convolutions at each iteration to calculate the value in every pixel $x \in \Omega$. Unfortunately, this is not possible with the expression of R that we have found before.

However, if $R^{\varphi}_{\epsilon,I}(x) = \sum_{y\in\Omega} w(||x-y||) q^\varphi(I(x), I(y))$ we can approximate the non-separable function q^φ by a polynomial p of order n on two variables

$$p(I(x), I(y)) = \sum_{j=0}^{n} \sum_{m=0}^{n-j} p_{m,j} I(x)^m I(y)^j = \sum_{j=0}^{n} \left(\sum_{m=0}^{n-j} p_{m,j} I(x)^m \right) I(y)^j = \sum_{j=0}^{n} f_j(I(x)) I(y)^j. \tag{25}$$

The coefficients of p are such that $p(I(x), I(y)) = \text{argmin}\, ||p - q||_2$, so p is the polynomial of order n that minimizes the quadratic distance from q. By substituting the approximated polynomial p to the real function q we get

$$\begin{aligned} R^{\varphi}_{\epsilon,I}(x) &= \sum_{y\in\Omega} w(||x-y||) \sum_{j=0}^{n} f_j(I(x)) I(y)^j = \sum_{j=0}^{n} f_j(I(x)) \left(\sum_{y\in\Omega} w(||x-y||) I(y)^j \right) \\ &= \sum_{j=0}^{n} f_j(I(x)) (w * I^j)(x), \qquad \text{where } f_j(I(x)) = \sum_{m=0}^{n-j} p_{m,j} I(x)^m. \end{aligned}$$

In the new algorithm the computation of the function $R^{\varphi}_{\epsilon,I}$ involves the calculation of the powers of I, n convolutions and the evaluation of $f_j, \forall\, j = 1, \ldots, n$. First of all we remark that powers can be fully calculated by making $(n-1)N$ multiplications, so the complexity of this operation is $\mathcal{O}(N)$ because n is a fixed parameter. Furthermore every convolution is computed in the frequency domain by using the formula $w * I^j = \mathcal{F}^{-1}(\mathcal{F}(w) \cdot \mathcal{F}(I^j))$, so it involves three FFT and N multiplication, for a total complexity of $\mathcal{O}(N \log N)$. Finally, considering that the possible functions q^{φ} are three, as the considered φ functions, and that $I(x)$ can take only the 255 discrete values in the set $\left\{\frac{1}{255}, \frac{2}{255}, \ldots, \frac{255}{255}\right\}$, we can precompute and store all the possible values taken by every f_j function. With this expedient this last operation has a linear cost $\mathcal{O}(N)$, given by the memory access to read the values. We can conclude that the total cost of the approximated algorithm is $\mathcal{O}(N \log N)$ as desired.

In Figure 1 we present some examples of results reported by the authors of the paper, obtained by the method presented in this report.

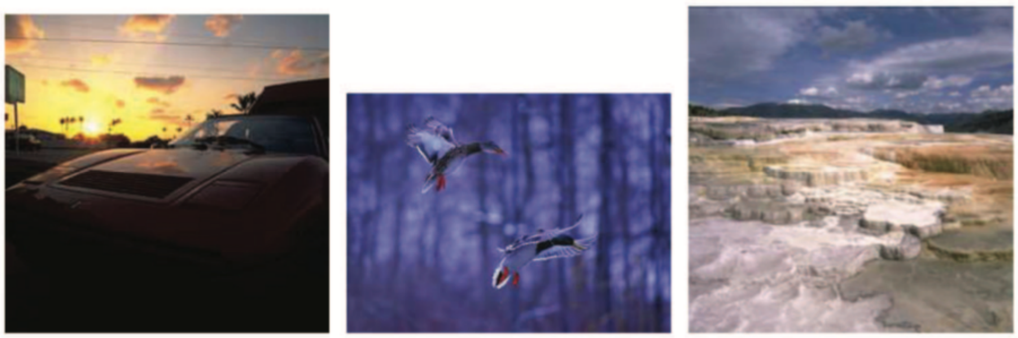

Figure 1. Three different original images showing different features to be enhanced: under exposure, color cast, over exposure.

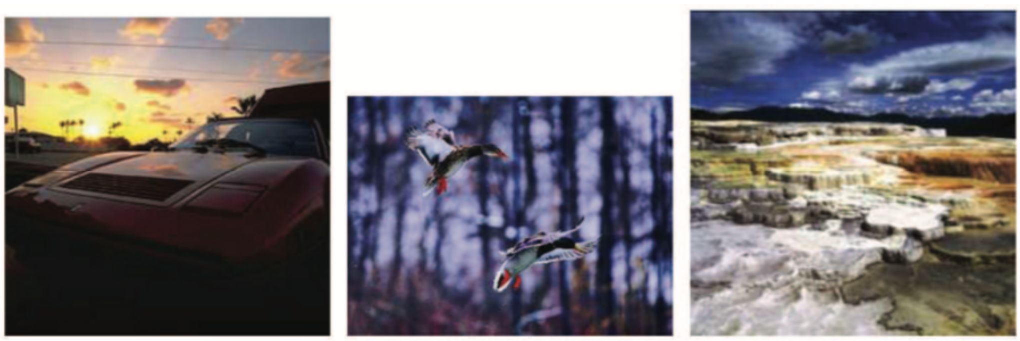

Figure 2. Outputs of the algorithm corresponding to the energy $E^{\mathrm{id}}_{w,\varepsilon}$.

6. APPLICATION TO FRESCO RESTORATION

We now propose a new application for this method. The problem in exam comes from the artistic world, in particular from the practice of restoration. Many ancient paintings, overall frescoes in churches, have some parts ruined by time and events. Sometimes an entire region is completely lost, we will call that part of the painting a blank.

One of the conservator's jobs is to find a suitable color to fill the blank. In fact the restorer cannot reproduce the lost part, he can just choose an uniform color to minimize the perception of the blank. The color has to be achromatic (a level of gray) and it should make the blank as imperceptible as possible. In the paper we have deduced the properties of a generic contrast function φ that can quantify the perceived contrast among a couple of points. The idea is to use this function to calculate the total perceived contrast C associated with an image and to find the color of the blank that minimizes that function

$$C^{\varphi}_{w}(I) = \sum_{x\in\Omega}\sum_{y\in\Omega} w(x,y)\varphi\left(\frac{\max(I(x),I(y))}{\min(I(x),I(y))}\right).$$

In practice blanks are identified by a mask, a grayscale image which takes the value 0 (black) where the original image should remain unchanged (absence of blanks) and a different non-zero value for each distinct blank.

Figure 3. A fresco by Tiepolo with two blanks artificially introduced and the associated mask.

We start by analyzing the case with only one blank and we consider each channel separately. We denote $\mathcal{L} \subset \Omega$ the set of pixels that corresponds to the blank and if we indicate the unknown color of the blank with λ we get

$$I(x) = \lambda \qquad \forall x \in \mathcal{L}. \tag{26}$$

If we fix the original image I outside the blank, the resulting total contrast is a function of λ

$$\begin{aligned}
C_w^{\varphi}(\lambda) &= \sum_{\substack{x\in\Omega \\ y\in\Omega}} w(x,y)\varphi\left(\frac{\max(I(x),I(y))}{\min(I(x),I(y))}\right) \\
&= \sum_{\substack{x\in\Omega\setminus\mathcal{L} \\ y\in\Omega\setminus\mathcal{L}}} w(x,y)\varphi\left(\frac{\max(I(x),I(y))}{\min(I(x),I(y))}\right) + \sum_{\substack{x\in\Omega\setminus\mathcal{L} \\ y\in\mathcal{L}}} w(x,y)\varphi\left(\frac{\max(I(x),I(y))}{\min(I(x),I(y))}\right) \\
&+ \sum_{\substack{x\in\mathcal{L} \\ y\in\Omega\setminus\mathcal{L}}} w(x,y)\varphi\left(\frac{\max(I(x),I(y))}{\min(I(x),I(y))}\right) + \sum_{\substack{x\in\mathcal{L} \\ y\in\mathcal{L}}} w(x,y)\varphi\left(\frac{\max(I(x),I(y))}{\min(I(x),I(y))}\right) \\
&= \sum_{\substack{x\in\Omega\setminus\mathcal{L} \\ y\in\Omega\setminus\mathcal{L}}} w(x,y)\varphi\left(\frac{\max(I(x),I(y))}{\min(I(x),I(y))}\right) + \sum_{\substack{x\in\Omega\setminus\mathcal{L} \\ y\in\mathcal{L}}} w(x,y)\varphi\left(\frac{\max(I(x),\lambda)}{\min(I(x),\lambda)}\right) \\
&+ \sum_{\substack{x\in\mathcal{L} \\ y\in\Omega\setminus\mathcal{L}}} w(x,y)\varphi\left(\frac{\max(\lambda,I(y))}{\min(\lambda,I(y))}\right) + \sum_{\substack{x\in\mathcal{L} \\ y\in\mathcal{L}}} w(x,y)\varphi(1) \\
&= \sum_{\substack{x\in\Omega\setminus\mathcal{L} \\ y\in\Omega\setminus\mathcal{L}}} w(x,y)\varphi\left(\frac{\max(I(x),I(y))}{\min(I(x),I(y))}\right) + 2\sum_{\substack{x\in\Omega\setminus\mathcal{L} \\ y\in\mathcal{L}}} w(x,y)\varphi\left(\frac{\max(I(x),\lambda)}{\min(I(x),\lambda)}\right) + \varphi(1)\sum_{\substack{x\in\mathcal{L} \\ y\in\mathcal{L}}} w(x,y)
\end{aligned} \tag{27}$$

where we have used the symmetry of functions $w, \max, \min$.

We want to solve the following problem

$$(\widetilde{P}): \text{ minimize } C_w^\varphi(\lambda) \text{ with } \lambda \in \left\{\frac{1}{255}, \dots, \frac{255}{255}\right\}.$$

We can remark that the first and the third term in equation (27) are independent on the variable λ, so the problem $(\widetilde{P})$ is equivalent to

$$(P): \text{ minimize } \sum_{\substack{x\in\Omega\setminus\mathcal{L}\\ y\in\mathcal{L}}} w(x,y)\varphi\left(\frac{\max(I(x),\lambda)}{\min(I(x),\lambda)}\right) \text{ with } \lambda \in \left\{\frac{1}{255}, \dots, \frac{255}{255}\right\}.$$

Finally, we have

$$\sum_{\substack{x\in\Omega\setminus\mathcal{L}\\ y\in\mathcal{L}}} w(x,y)\varphi\left(\frac{\max(I(x),\lambda)}{\min(I(x),\lambda)}\right) = \sum_{x\in\Omega\setminus\mathcal{L}} \left(\sum_{y\in\mathcal{L}} w(x,y)\right)\varphi\left(\frac{\max(I(x),\lambda)}{\min(I(x),\lambda)}\right),$$

so we can separate the calculation of $s(x) = \sum_{y\in\mathcal{L}} w(x,y)$, which is independent of λ, from the resolution of the problem. The computation of $s(x)$ is the most time-consuming part of the calculation, but we can precompute this quantity for each pixel outside the blank because it depends just on the shape and the position of the blank.

We have deduced our functional, now we have to find a suitable computational way to find its minimum. The input of the algorithm is given by the original image, the associated mask and all the parameters that define w and φ. In order to solve the problem (P), we could use a gradient descent method, but this is inefficient because we know that the number of possible values for λ is 255 and we don't know how many iterations would be necessary to reach the minimum by using a gradient descent strategy. If this number exceeds 255 the computation is more expensive and less precise. So the most simple and efficient way to minimize the function is to test each possible value of λ and to choose the point where the function is minimal.

Until now we have considered just a single channel, in the case of color images the operation described are repeated in every channel. The additional effort is almost negligible because we can calculate $s(x)$ once and use it for every channel. The result of this algorithm is the uniform color that makes the blank the least possible perceivable, but a restorer is obliged to choose a level of gray to fill the blank. In this case it is sufficient to call the procedure on the grayscale image associated with the original one.

If we do not have the precomputed value of $s(x)$ this method is too slow, so we have looked for ways to approximate this function to reduce the time. First, we can remark that $s(x)$ is a continuous function because of the continuity of w. So it is possible to calculate the value of $s(x)$ just on a grid of points and set the value of a point as that of the nearest neighbor. This method saves a lot of time and we have observed that the results are unchanged with respect to the ones obtained from the direct computation.

Moreover we can find another approximation that makes the computation also cheaper. We can notice that, if the blank is a circle, $s(x)$ depends only on the distance of x from the center of the circle. This fact is true just for a circle because it is the only geometry invariant under rotations. From this remark it is possible to deduce a second approximated algorithm. We check if the blank can be reasonably approximated as a circle and we estimate its center O and radius r. The approximated center is the centroid of the blank and the radius r is the solution of the problem

$$(R): \text{ maximize } q(r) = \frac{\beta(r)}{\alpha(r)} + \frac{\beta(r)}{A},$$

where $\beta(r)$ is the number of pixels of the blank covered by the circle of radius r, $\alpha(r)$ represents the number of pixels covered by the circle of radius r and A is the number of pixels of the blank. In practice the problem (R) is solved by using the Golden section search algorithm and the circle approximation is accepted if $q(r)$ exceed a certain fixed threshold $\eta \in [0,2]$. If the blank is well approximated by a circle we compute all the values of $s(d)$ for all $d \in \mathbb{N} \cap [r, d_{\text{MAX}}]$ where d_{MAX} is the maximum distance of a pixel from the center O. To finish we pose

$$s(x) = s(\text{round}(||x - O||)) \quad \forall x \in \Omega \setminus \mathcal{L}.$$

In order to extend the algorithm for multiple blanks it is sufficient not to consider the other blanks during the computation. We have decided not to take into account the influence among the blanks because if the painting presents two or more blanks close together, the user can call the procedure with a mask that considers them as a unique blank. On the other hand, if the blanks are far apart their reciprocal influence is negligible. If the painting has p different blanks $\mathcal{L}_1, \ldots, \mathcal{L}_p$ and we denote $\mathcal{L} = \cup_{i=1}^{p} \mathcal{L}_i$ the problem (P) is composed by p sub-problems (P_i) where $i \in 1, \ldots, p$

$$(P_i): \text{ minimize } \sum_{x \in \Omega \setminus \mathcal{L}} \left(\sum_{y \in \mathcal{L}_i} w(x,y) \right) \varphi \left(\frac{\max(I(x), \lambda_i)}{\min(I(x), \lambda_i)} \right) \text{ with } \lambda_i \in \left\{ \frac{1}{255}, \ldots, \frac{255}{255} \right\}.$$

We can solve each sub-problem separately to get the final solution.

Experiments show that there is no difference in the results obtained from the direct computation and from the two approximated algorithms. The real difference comes from the choice of w and φ. In Figs. 4, 5, 6 we show some tests that we have performed with the described approximated procedure, they differ with respect to the parameter φ and w selected

7. TESTS

8. CONCLUSIONS AND PERSPECTIVES

In this paper, we have revised and applied the mathematical model for perceptually-inspired color correction proposed in[1] to the problem of tone selection to fill lacunae in frescoes. The computational model that we have proposed is based on the minimization of an energy functional that represent the perceived local contrast of a lacuna with respect to the surrounding area. The solution of the variational problem is then the gray level, or the intensity level in each separate chromatic channel, which makes the lacuna least perceptible.

Perceptual tests are still needed to properly select the two parameters of the proposed method: a spatially local weight function w and a monotonically increasing function φ. Moreover, the digital solution provided by the computational algorithm must be transduced into the corresponding pigment that will be used to fill the lacuna.

It is important to stress that, even after these steps, the proposed method can only be considered a useful tool to facilitate the restoration task and it cannot substitute the role of an expert in art restoration. In fact, her/his role is fundamental to embed the problem into the historical and cultural setting proper to the painting or fresco, which may lead to a modification of the color provided by our model.

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[2] Gonzales, R. and Woods, R., [*Digital image processing*], Prentice Hall (2002).

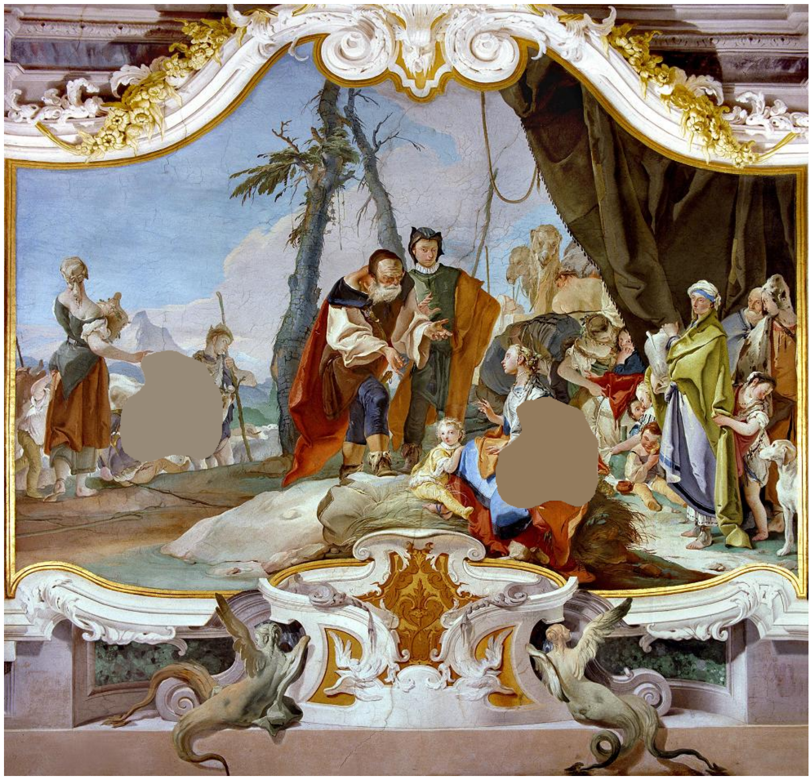

Figure 4. Result of the algorithm with $\varphi = \log$ and w gaussian ($\sigma = 10$).

Figure 5. Result of the algorithm with $\varphi = \mathcal{M}$ and w linear ($\alpha = 5$).

Figure 6. Result of the algorithm with $\varphi = \text{id}$ and w gaussian ($\sigma = 5$).